# Explicit solutions and linearisation of certain non-linear evolution equations-bilinear transformation method



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COMMENT

# Explicit solutions and linearisation of certain non-linear evolution equations—bilinear transformation method

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**Abstract.** Considering the complex Burgers, Cheng chemical reaction and Liouville equations, we show that through Hirota's bilinearisation method solutions can be obtained straightforwardly. We also show that these systems are exactly linearisable.

It is well known that Hirota's bilinear transformation method (Hirota 1971, 1976, Matsuno 1984) is the most convenient one, among direct methods, of finding solutions to non-linear evolution equations. In this comment we wish to apply this bilinearisation procedure to linearise exactly and obtain explicit solutions of the complex Burgers, Cheng chemical reaction and Liouville equations in a natural way.

We first consider the complex Burgers equation in the form

$$u_t + \mathrm{i} u_{xx} + 4uu_x = 0. \tag{1}$$

Recently, Bruschi and Ragnisco (1985) have proved that (1) admits an infinite number of constants of motion in involution. By the dependent variable transformation

$$u = g/f \tag{2}$$

(1) can be rewritten in the bilinear form

$$(D_t + \mathrm{i} D_x^2) gf = 0 \tag{3}$$

and

$$4D_x gf - \mathrm{i} D_x^2 ff = 0. \tag{4}$$

Here, $D_x$ and $D_t$ are the usual bilinear operators defined by

$$D_x^m D_t^n ab = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, t) b(x', t') \bigg|_{\substack{x = x' \\ t = t'}}. \tag{5}$$

Since $g = \frac{1}{2}\mathrm{i} f_x$ is a solution of (4), (3) becomes

$$(D_t + \mathrm{i} D_x^2) f_x f = -2\mathrm{i} D_x (f_t - \mathrm{i} f_{xx}) f = 0.$$

Thus

$$f_t - \mathrm{i} f_{xx} = 0 \tag{6}$$

holds, and thereby an exact linearisation of (1) is effected. From the above analysis, it is clear that if $f$ is a solution of the Schrödinger equation (6), then $g = \frac{1}{2}\mathrm{i} f_x$ in (2) immediately implies the celebrated Cole-Hopf transformation in the complex form, which solves the complex Burgers equation (1).

To obtain the explicit solutions of (1), we expand $f$ and $g$ as a formal power series in the parameter $\varepsilon$:

$$f = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n \qquad g = \sum_{n=0}^{\infty} \varepsilon^n g_n. \tag{7}$$

Truncating the series in (7) at $n = 1$ and 2, we can easily obtain one and two solitary wave solutions of (1), respectively, in the form

$$u = \frac{\exp(\eta_1)}{1 - (2\mathrm{i}/p_1)\exp(\eta_1)} \tag{8}$$

and

$$u = \frac{\exp(\eta_1) + \exp(\eta_2)}{1 - (2\mathrm{i}/p_1)\exp(\eta_1) - (2\mathrm{i}/p_2)\exp(\eta_2)} \tag{9}$$

where $\eta_j = p_j x - \mathrm{i}p_j^2 t$, $j = 1, 2$.

It is of interest to note that the above type of solutions (8) and (9) exist for the class of equations $L_q K = \text{constant} \times K^{N-1} K_x$, where $N \geqslant 2$ and $L_q$ is the $q$th-order linear differential operator in $1+1$ dimensions, as noted by Cornille (1982).

Next, we consider Cheng's equation (1984)

$$U_x = -aUV \qquad V_t = bU_x \tag{10}$$

describing the dynamics of photosensitive molecules when a light beam passes through them. In (10), $a$ and $b$ are the absorption and the proportionality constants, respectively. Also, the variables $U$ and $V$ are the light intensity and density of the molecules, respectively.

Introducing the transformations

$$V = (1/a)(\log f)_x \qquad U = g/f \tag{11}$$

in (10), we obtain the bilinearised equations

$$D_x g f + g f_x = 0 \tag{12}$$

and

$$(1/a)(D_x D_t ff) - bD_x gf = 0. \tag{13}$$

In addition, we make the transformations

$$f = 1 + \exp\theta \qquad g = \exp\tilde{\theta} \tag{14}$$

in (12) and (13), and find that

$$g = (-1/ab)\theta_t. \tag{15}$$

But from (12), we observe that $g_x = 0$ and hence (15) leads to the linear wave equation $\theta_{xt} = 0$. Utilising the solution of the linear wave equation $\theta = P(x) + Q(t)$, where $P(x)$ and $Q(t)$ are arbitrary, in (11) we obtain

$$\begin{aligned} U &= -(1/ab)Q_t[1 + \exp(P(x) + Q(t))]^{-1} \\ V &= (1/ab)P_x \exp(P(x) + Q(t))[1 + \exp(P(x) + Q(t))]^{-1} \end{aligned} \tag{16}$$

which is a solution of (11). The general solution of Cheng's equation (10) follows by setting $g(x) = \exp(P(x))$, $h(t) = \exp(-Q(t))$ (Tamizhmani and Lakshmanan 1986).

Finally, we consider the well known Liouville equation (Rogers and Shadwick 1982 and references therein)

$$u_{xt} - e^{u} = 0. \tag{17}$$

On using

$$u = \log V \tag{18}$$

(17) can be rewritten as

$$VV_{xt} - V_x V_t - V^3 = 0. \tag{19}$$

With the dependent variable transformation

$$V = g/f^2 \tag{20}$$

(19) can be written in the following form:

$$D_x D_t gg = 0 \qquad f_{xt} = 0 \tag{21a}$$

and

$$2f_x f_t = g. \tag{21b}$$

Solving (21), we find that

$$f(x, t) = A(x) + B(t) \tag{22}$$

and

$$g(x, t) = 2A_x B_t \tag{23}$$

where $A(x)$ and $B(t)$ are arbitrary functions. From (22), (23) and (20), (18) becomes

$$u = \log\left(\frac{2A_x B_t}{(A+B)^2}\right) \tag{24}$$

which is the well known general solution of (17).

Thus we have proved that the Hirota bilinearisation procedure straightforwardly gives the general solutions of all the above non-linear evolution equations, exactly linearising them in this process.

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